Vibration Analysis of Composite Beam

A Thesis Submitted in Partial Fulfilment
of the Requirements for the Award of the Degree of

Master of Technology
in
Machine Design and Analysis

by

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Under the guidance of

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June 2013

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CERTIFICATE

This is to certify that the thesis entitled "**Vibration Analysis of Composite Beam**" by Hemanshu, submitted to the National Institute of Technology (NIT), Rourkela for the award of Master of Technology in **Machine Design and Analysis**, is a record of bona fide research work carried out by him in the Department of Mechanical Engineering, under our supervision and guidance.

I believe that this thesis fulfills part of the requirements for the award of degree of Master of Technology. The results embodied in the thesis have not been submitted for the award of any other degree elsewhere.

Place: Rourkela

Date:

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ABSTRACT

Beams are the basic structural components. When they are made laminated composites their strength to weight ratio increases. They can be used for different application just by changing the stacking sequence in the laminate with the same weight and dimensions. So this requires a complete analysis of laminated composite beams. They are used in a variety of engineering applications such as airplane wings, helicopter blades, sports equipment's, medical instruments and turbine blades. An important element in the dynamic analysis of composite beams is the computation of their natural frequencies and mode shapes. It is important because composite beam structures often operate in complex environmental conditions and are frequently exposed to a variety of dynamic excitations. In this research work first order shear deformation theory is used for vibration analysis of composite beams. A dynamic analysis is carried out which involves finding of natural frequencies and mode shapes for different L/H ratios and different stacking sequences. Finally the non-dimensional natural frequencies of the beam are calculated by using MATLAB and ANSYS model of corresponding composite beam.

CONTENTS

CERTIFICATE **i**

ACKNOWLEDGMENT **ii**

ABSTRACT **iii**

CONTENTS **iv**

LIST OF TABLES **vi**

LIST OF FIGURES **vii**

LIST OF ACRONYM **viii**

CHAPTER 1: Introduction **1**

1.1 Introduction of composite materials and structures 1

1.2 Literature Review 2

1.3 Objective 9

1.4 Thesis Organization 9

CHAPTER 2: Finite Element Model for FSDT **10**

2.1 First order shear deformation theory 10

2.2 Elasticity Matrix (D) 11

2.3 Strain-Displacement Matrix 14

2.4 Isoparametric element 15

2.5 Mass Matrix 16

2.6 Element stiffness matrix 17

2.7 Hamilton's principle 17

CHAPTER 3: Finite Element Model for HSDT 20

3.1 Higher order shear deformation theory 20

3.2 Strain-Displacement Matrix 24

3.3 Mass Matrix 24

3.4 Element Stiffness Matrix 25

CHAPTER 4: Result and Discussion 27

4.1 Problem 27

4.2 Results and discussion 28

CHAPTER 5: Conclusion and Future work 35

APPENDIX 36

REFERENCES 40

LIST OF TABLES

Table 1: Non dimensional fundamental frequencies λ for clamped-free unidirectional composite beam. 32

Table 2: Non dimensional fundamental frequencies (λ) for composite beam (l/h=60). 33

Table 3: Non dimensional fundamental frequencies (λ) for composite beams (l/h=5). 33

LIST OF FIGURES

Figure 1: Comparison of composite material properties with steel and aluminium. 2

Figure 2: Geometry of laminated composite beam. 11

Figure 3: Nine node isoparametric element. 16

Figure 4: Discretized composite beam. 17

Figure 5: Natural Frequency for diff. values of theta and constant (l/h=5). 34

Figure 6: Natural Frequency for different l/h ratio for (0/90/0). 34

Figure 7: Natural Frequency for different stacking sequence for (l/h=5) and CC BC. 35

Figure 8: Natural Frequency for different boundary conditions for (0/90/0) and (l/h=60). 36

Figure 9: 1st mode shape for clamped-clamped BC. 37

Figure 10: 2nd mode shape for clamped-clamped BC. 37

Figure 11: 3rd mode shape for clamped-clamped BC. 37

Figure 12: 4th mode shape for clamped-clamped BC. 37

Figure 13: 1st mode shape for clamped-free BC. 38

Figure 14: 2nd mode shape for clamped-free BC. 38

Figure 15: 3rd mode shape for clamped-free BC. 38

Figure 16: 4th mode shape for clamped-free BC. 38

LIST OF ACRONYMS

List of Acronyms

FSDT	**:** First Order Shear Deformation Theory
HSDT	**:** Higher Order Shear Deformation Theory
2D	: Two Dimensional
3D	: Three Dimensional
FE	**:** Finite Element
LCB	**:** Laminated Composite Beam
CLPT	**:** Classical Laminated Plate Theory
GPa	: Giga Pascal
BC'S	: Boundary Conditions
DOF	: Degree of Freedom
CC	: Clamped-Clamped
CF	: Clamped-Free
SS	: Simply Supported

INTRODUCTION

Since many years ago, the combination of different materials has been used to achieve better performance requirements. As an example of that the Sumerians in 4000 B.C. used to add straw to the mud to increase the resistance of the bricks. Although the benefits brought by the composite materials are known for thousands of years, only a few years ago the right understanding of their behavior as well as the technology for designing composites started to be developed. The airplane F111 was one of the first models to incorporate composite technology. Also airplane Boeing 767 has 2 tons in composite materials. The possibility to combine high strength and stiffness with low weight has also got the attention of the automobile industry: the Ford Motor Company developed in 1979 a car with some components made from composite materials. The prototype was directly 570 kg lighter than the same version in steel, the transmission shaft had a huge reduction of 57% of its original weight. More recently, Chrysler developed a car completely based on composite materials, known as CCV (Composite Concept Vehicle). Besides these examples in the automobile and aeronautical industry, the applications of composite materials have been enlarged, including now areas as the sporting goods, civil and aerospace construction and in medical field. In order to have the right combination of material properties and in service performance, the static and dynamic behavior is one of the main points to be considered.

Thus, the main objective of this work is to contribute for a better understanding of vibration analysis of components made from composite materials, specifically for the case of beams. In the present investigation I have used First Order Shear Deformation Theory (FSDT) and Higher Order Shear Deformation Theory (HSDT) to analyze the composite beams. In order to investigate the influence of the stacking sequence, l/h ratio and boundary conditions on natural frequencies and mode-shapes of the composite beam. A MATLAB program is written for both theories used in the investigation and also an ANSYS model is made of composite beam and results are compared with the past author works. Fig. 1 shows comparison of steel, aluminium and composite material (S-Glass) on the basis of weight, thermal expansion, stiffness and strength.

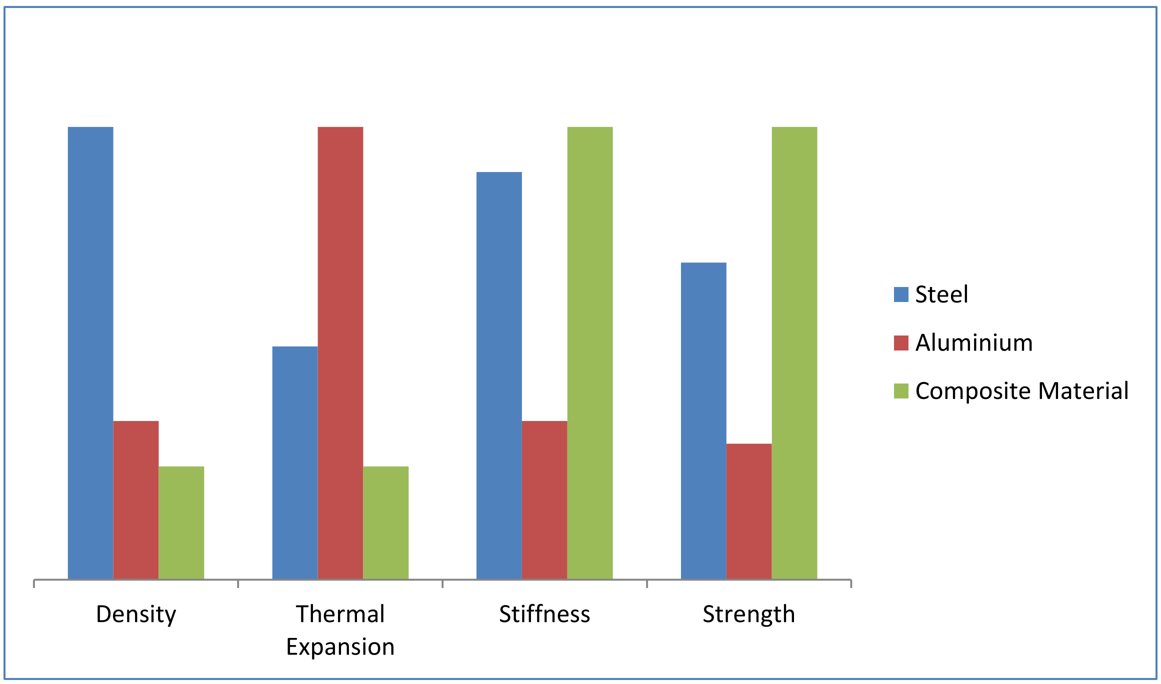


Fig. 1 Comparison of composite material properties with steel and aluminium.

LITERATURE REVIEW

2.1 Literature Review

Composite materials are widely used in structures, especially in aircraft and spacecraft, due to their high strength-to-weight and stiffness-to weight ratios. Their behavior is designed according to their usage, so that their advantages are fully utilized. Laminated composite beams are normally analyzed by means of energy methods and classical laminate plate theory (CLPT). This literature provides vibration analysis of laminated composite beams by first and higher order shear deformation theories.

Maiti & Sinha [1] used higher order shear deformation theory for the analysis of composite beams. Nine noded isoparametric elements are used in the analysis. Natural frequencies of composite beam are compared for different stacking sequences, different (l/h) ratios and different boundary conditions. They had shown that natural frequency decreases with an increase in ply angle and a decrease in (l/h) ratio.

Jafari and Ahmadian [2] had done free vibration analysis of a cross-ply laminated composite beam on Pasternak Foundation. The model is designed in such a way that it can be used for single-stepped cross-section. For the first time to-date, the same analysis was conducted for a single-stepped LCB on Pasternak foundation. Stiffness and mass matrices of a cross-ply LCB on Pasternak foundation using the energy method are computed.

Raciti and Kapania [3] collected a report of developments in the vibration analysis of laminated composite beams. Classical laminate plate theory and first order shear deformation theory are used for analysis. The assumption of displacements as linear functions of the coordinate in the thickness direction has proved to be inadequate for predicting the response of thick laminates.

Teboub and Hajela [4] approved the symbolic computation technique to analyze the free vibration of generally layered composite beam on the basis of a first-order shear deformation theory. The model used considering the effect of poisson effect, coupled extensional, bending and torsional deformations as well as rotary inertia.

Bassiouni [5] proposed a finite element model to investigate the natural frequencies and mode shapes of the laminated composite beams. The FE model needed all lamina had the same lateral displacement at a typical cross-section, but allowed each lamina to rotate to a different amount from the other. The transverse shear deformations were included.

Banerjee [6] has investigated the free vibration of axially laminated composite Timoshenko beams using dynamic stiffness matrix method. This is accomplished by developing an exact dynamic stiffness matrix of a composite beam with the effects of axial force, shear deformation and rotatory inertia taken into account. The effects of axial force, shear deformation and rotatory inertia on the natural frequencies are demonstrated. The theory developed has applications to composite wings and helicopter blades.

Yuan and Miller [7] derived a new finite element model for laminated composite beams. The model includes sufficient degrees of freedom to allow the cross-sections of each lamina to deform into a shape which includes up through cubic terms in thickness co-ordinate. The element consequently admits shear deformation up through quadratic terms for each lamina but not interfacial slip or delamination.

Krishnaswamy [8] have studied the free vibration of LCBs including the effects of transverse shear and rotary inertia. Dynamic equations governing the free vibration of laminated composite beams are developed using Hamilton's principle. Analytical solutions are obtained by the method of Lagrange multipliers. Natural frequencies and mode shapes of clamped-clamped and clamped-supported composite beams are presented to demonstrate the efficiency of the methodology.

Chandrashekhara [9] have presented exact solutions for the vibration of symmetrically LCBs by first order shear deformation theory. Rotary inertia has been included but Poisson effect has been neglected and demonstrated the effect of shear deformation, material anisotropy and boundary conditions on the natural frequencies.

Subramanian [10] has investigated free vibration analysis of LCBs by using two higher order displacement based shear deformation theories and finite element. Both theories assume a variation of in-plane and transverse displacements in the thickness coordinates of the beam respectively. Results indicate application of these theories and finite element model results in natural frequencies with higher accuracy.

A study of literature by Ghugal and Shimpi [11] indicates that the research work dealing with flexural analysis of thick beams using refined hyperbolic, trigonometric and exponential shear deformation theories is very scant and is still in early stage of development.

Pagano NJ [12] investigated the limitation of CLPT by comparing solutions of several specific boundary value problems in this theory to the corresponding theory of elasticity solutions. The general class of problems treated involves the geometric configuration of any number of isotropic or orthotropic layers bonded together and subjected to cylindrical bending. In general it is found that conventional plate theory leads to a very poor description of laminate response at low span-to-depth ratios, but converges to the exact solution as this ratio increases. The analysis presented is also valid in the study of sandwich plates under cylindrical bending.

Yildiz and Sarikanat [13] developed a finite-element analysis program to analyze multi-layer composite beams and plates. The arithmetic average and weighted average method were developed. By considering different loading conditions, one of the averaging methods was used. The effects of both averaging methods on the results were investigated. On comparing the obtained results with the analytical solutions, here we can see that both methods are giving matching results for certain types of loading.

Oral [14] developed a shear flexible finite element for non-uniform laminated composite beams. He tested the performance of the element with isotropic and composite materials, constant and variable cross-sections, and straight and curved geometries.

A method proposed by Hurty [15] enabled the problem to be broken up into separate elements and thus considerably reduced its complexity. His method consisted of considering the structure in terms of substructures and was called as sub structuring. Essentially, the method required the derivation of the dynamic equations for each component and these equations were then connected mathematically by matrices which represent the physical displacements of interface connection points on each component. In this way, one large Eigen value problem is replaced by several smaller ones. There are applications where alternative justifications are valid, for example where the results of independent analysis of individual structural modules are to be used to predict the dynamics of an assembled structure.

Ergatoudis and Zienkiewicz [16] describes the theory of a new family of Isoparametric elements for use in two-dimensional situations. The possibilities of improvement of approximation are thus confined to devising alternative element configurations and developing new shape functions. An obvious improvement is the addition of a number of nodal points along the sides of such elements thus permitting a smaller number of variables to be used for solution of practical problems with a given degree of accuracy. Examples illustrating the accuracy improvement are included.

Bhimaraddi and Chandrashekhara [17] had done modeling of laminated beams considering a systematic reduction of the constitutive relations of the three-dimensional anisotropic body. The basic equations of the beam theory here are those of the parabolic shear deformation theory. Numerical results for natural frequencies and the Euler buckling load have been presented using the modeling of the constitutive relations and those of the conventional type modeling. It has been observed from the numerical results that the two approaches differ little in the case of cross-ply laminates but there exists a considerable difference (by a multiple factor of 3 in some cases) in the case of angle-ply laminates.

Spadea and Zinno [18] analysis shows that the finite element approach requires more computer equipment and engineer expertise but enables a more general and consistent analysis. Nevertheless, if a more realistic assessment of the behavior of this kind of structure is carried out, the cost is reduced and a higher safety factor usually gained.

Banerjee and Williams [19] used the explicit stiffness expressions, as opposed to the numerical computation of the dynamic stiffness matrix by inversion, is illustrated by comparing elapsed CPU times. The application of the derived dynamic stiffness matrix to calculate the natural frequencies and mode shapes of bending-torsion coupled composite beams uses the Wittrick-Williams algorithm.

Yong-Bae and Ronald [20] discussed a refined beam theory based on sub laminate linear zig - zag kinematics and a new two-dimensional finite element based theory is developed. The new CO element contains four nodes of which each has only three engineering degrees of freedom - two translations and one rotation. The element is shown to be accurate, simple to use and compatible with the requirements of commercial finite-element codes.

Akavci, Yerli and Dogan [21] used classical theory of plates (CPT), it is assumed that plane sections initially normal to the mid surface before deformation remain plane and normal to that surface after deformation. As a result of neglecting transverse shear strains. However, there are non-negligible shear deformations occurring in thick and moderately thick plates. This theory gives inaccurate results for laminated plates. So the transverse shear deformations should be taken into account in the analysis of composite structures.

Sayyad [22] compared refined beam theories for the free vibration analysis of thick beams by taking into account transverse shear deformation effect. This theory involves exponential, sinusoidal, parabolic and hyperbolic functions in terms of thickness coordinates to include transverse shear deformation effect. In this theory the numbers of unknowns are same as that of FSDT. The governing differential equations and boundary conditions are obtained by using the principle of virtual work. And the results of bending and thickness shear mode frequencies for simply supported beam are presented and discussed critically with those of other theories.

Mohammed, Goda and Galal [23] used a combined finite element and experimental approach to characterize the vibration behavior of composite beams. Here glass fiber is used as reinforcement in the form of bidirectional fabric and polyester resin as matrix for the beam. Experimental dynamic tests are carried out using specimens with different fiber orientations and stacking sequences. From the results, the influence of fiber orientations on the flexural natural frequencies is investigated. Also, the finite element software ANSYS is used to validate the results obtained from these experiments.

Kant and Swaminathan [24] found the solution of laminated composite plates using higher order shear deformation theory. Natural frequencies are compared for various (E_1/E_2), (l/h) ratios and by varying stacking sequence. For laminated composite plates the solutions of this higher order refined theory are found to be in good agreement with three-dimensional elasticity solutions.

So, In order to have the right combination of material properties and in service performance, the static and dynamic behavior is one of the main points to be considered. Thus, the main objective of this work is to contribute for a better understanding of vibrational analysis of composite beams. In the present investigation first order shear deformation theory (FSDT) and higher order shear deformation theory (HSDT) are used to analyze composite beam. In order to investigate the influence of the stacking sequence, (l/h) ratio and different boundary conditions on natural frequencies and mode-shapes of the composite beam, MATLAB program is written for both theories. An ANSYS software APDL program is written for the composite beam.

2.2 Objective

Based on the literature survey and the scope outlined in the previous sub-section, the following objectives are framed for the present research work

- Development of a finite element model for calculating natural frequencies of the composite beam using first order shear deformation theory (FSDT).
- Development of a finite element model for calculating natural frequencies of the composite beam using higher order shear deformation theory (HSDT).
- MATLAB and ANSYS finite element software program for calculating natural frequencies and mode shapes of the composite beam.

2.3 Thesis Organization

This thesis has six chapters organized as

- Chapter 1 provides an introduction about composite materials and their benefits for using in beams and why the vibration analysis is essential for composite beams.
- Chapter 2 gives a comprehensive review of previous research work.
- Chapter 3 provides a full description of developing a finite element model for calculating natural frequencies of composite beam by first order shear deformation theory (FSDT).
- Chapter 4 provides a full description of developing a finite element model for calculating natural frequencies of composite beam by higher order shear deformation theory (HSDT).
- Chapter 5 provides numerical results obtained for the laminated composite beam (LCB) and their discussion.
- Chapter 6 provides conclusions and future work.

FINITE ELEMENT MODEL FOR FSDT

3.1 First order shear deformation theory

Reissner and Mindlin [29, 32] is a well-known theory for the analysis of composite structures. This theory is also known as first order shear deformation theory (FSDT) and takes the displacement field as linear variations of mid plane displacements. Here the relation between the resultant shear forces and the shear strains is affected by the shear correction factors. This theory has some advantages as its simplicity and low computational cost. The geometry of a laminated composite beam is shown in fig. 2.

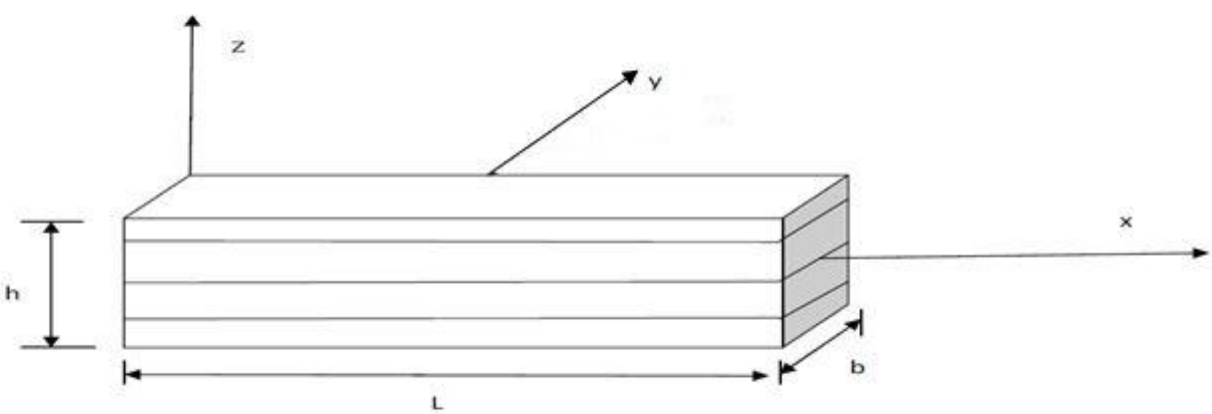


Fig. 2 Geometry of laminated composite beam.

Here,

L, b and h are length, breadth and thickness of the laminated composite beam.

To approximate a 3D elasticity problem into a 2D beam problem, the displacement functions u, v and w of the laminate at a point x, y and z are expanded in a Taylor series in terms of thickness co-ordinate as [1]. Where the displacement functions u, v and w can be written as

$u = u_0(x, y) + z\,\Theta_x(x, y)$,

$v = v_0(x, y) + z\,\Theta_y(x, y)$ and

$w = w_0(x, y)$ respectively. (1)

Here u, v and w are in-plane and transverse displacement components at any point along the x, y and z-axis in the laminate. The displacements along x, y and z axis are u_0, v_0 and w_0 respectively. Also middle plane slopes are θ_x and θ_y.

3.2 Elasticity Matrix (D)

Strain – displacement relations for the lamina are

$$\varepsilon_x = \varepsilon_{x_0} + z\kappa_x, \varepsilon_y = \varepsilon_{y_0} + z\kappa_y,$$
$$\gamma_{xy} = \varepsilon_{xy_0} + z\kappa_{xy}, \gamma_{yz} = \varepsilon_{yz_0}, \gamma_{xz} = \varepsilon_{xz_0}. \qquad (2)$$

Stress-strain relations for a lamina with respect to the fiber-matrix co-ordinate axis (1, 2, 3) are

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \\ \tau_{23} \\ \tau_{13} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{22} & 0 & 0 & 0 \\ 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{13} \end{Bmatrix}. \qquad (3)$$

Here (σ_1, σ_2, τ_{12}, τ_{23}, τ_{31}) are the stresses and (ε_1, ε_2, γ_{12}, γ_{23}, γ_{13}) are the strain components corresponding to the lamina co-ordinates (1, 2, 3). C_{ij} is the compliance matrix with respect to lamina axis (1, 2, 3) and is defined in appendix A. In laminate co-ordinates (x, y, z) the stress-strain relations for the lamina are given as

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{14} & 0 & 0 \\ Q_{12} & Q_{22} & Q_{24} & 0 & 0 \\ Q_{14} & Q_{24} & Q_{44} & 0 & 0 \\ 0 & 0 & 0 & KQ_{55} & KQ_{56} \\ 0 & 0 & 0 & KQ_{56} & KQ_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} . \tag{4}$$

Here (σ_x, σ_y, τ_{xy}, τ_{yz}, τ_{zx}) are the stresses and (ε_x, ε_y, γ_{xy}, γ_{yz}, γ_{zx}) are the strain components corresponding to the laminate co-ordinates (x, y, z). Q_{ij}'s are the transformed elasticity constants and are defined in appendix A.

Here K is shear correction factor [34]. It appears as a coefficient in the expression for the transverse shear stress resultant to consider the shear deformation effect with good approximation. Due to low shear modulus in multilayered plate and shell finite elements, there is an appreciable constant shear deformation. As the transverse shear stresses are zero at top and bottom faces and maximum at neutral axis. So the constant shear distribution across the thickness causes a decrease in accuracy. So, one has to multiply shear correction factor with transverse shear stress components. Numerical value of K depends upon Poisson's ratio, shape of the cross section and ply angle for the composite beam. Shear correction factor considers the effect of extension-shear coupling. Shear correction factor make neutral axis of the beam coincide with its geometric axis. From here the elasticity matrix [D] is derived as

$$[D]=[T]^{'}[Q_{ij}][T]. \tag{5}$$

Here [T] is the thickness co-ordinate matrix. So, D-matrix for FSDT can be written as

$$\left[D\right]=\begin{bmatrix} A_{ij} & B_{ij} & 0 \\ B_{ij} & D_{ij} & 0 \\ 0 & 0 & AA_{ij} \end{bmatrix}. \tag{6}$$

Here

$$\left(A_{ij},B_{ij},D_{ij}\right)=\int_{-h/2}^{h/2} Q_{ij}\left(1,Z,Z^{2}\right)dZ \rightarrow i,j=1,2,4 \text{ and}$$

$$\left(AA_{ij}\right)=\int_{-h/2}^{h/2} Q_{ij}\left(1\right)dZ \rightarrow i,j=5,6. \tag{7}$$

Here [A_{ij}] is extensional stiffness matrix, [B_{ij}] is stretching-bending coupling matrix and [D_{ij}] is flexural stiffness matrix.

3.3 Strain-Displacement Matrix

The strain-displacement relation is

$$\{\varepsilon\} = [L]\{\delta\}. \tag{8}$$

Here [L] is operator matrix, {ε} is the in-plane strain matrix and {δ} is the displacement at any point on the mid-plane of the element. The strain-displacement matrix [B] is obtained by multiplying operator matrix by shape functions as

$$[B] = [L] * N_r \rightarrow r = 1,2,3,4...9. \tag{9}$$

So, the B-matrix for FSDT can be given by

$$[B] = \begin{bmatrix} \frac{\delta N_r}{\delta x} & 0 & 0 & 0 & 0 \\ 0 & \frac{\delta N_r}{\delta y} & 0 & 0 & 0 \\ \frac{\delta N_r}{\delta y} & \frac{\delta N_r}{\delta x} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\delta N_r}{\delta x} & 0 \\ 0 & 0 & 0 & 0 & \frac{\delta N_r}{\delta y} \\ 0 & 0 & 0 & \frac{\delta N_r}{\delta y} & \frac{\delta N_r}{\delta x} \\ 0 & 0 & \frac{\delta N_r}{\delta y} & 0 & N_r \\ 0 & 0 & \frac{\delta N_r}{\delta x} & N_r & 0 \end{bmatrix}. \tag{10}$$

3.4 Isoparametric element

Isoparametric elements are used for mapping from one co-ordinate system to other co-ordinate system [28, 31, 34]. Natural co-ordinate system (ζ, η, ξ) is used while problem domain is in physical co-ordinate system (x, y, z).

The purpose of using isoparametric element is that it is difficult to represent curved boundaries by straight edges finite elements. So for such problems one can use isoparametric elements. In isoparametric elements number of nodes defining geometry is equal to number of nodes defining displacements.

In this study a nine-noded isoparametric element is used (fig. 3). The shape functions for a nine-noded quadrilateral isoparametric element are

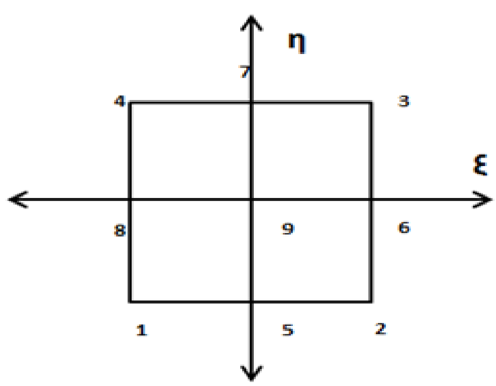


Fig. 3 Nine node isoparametric element.

$$
\begin{aligned}
N_i &= 1/4(\xi^2+\xi_i\xi)(\eta^2+\eta_i\eta); \rightarrow i=1,2,3,4 \\
N_i &= 1/2(1-\xi^2)(\eta^2+\eta_i\eta); \rightarrow i=5,7 \\
N_i &= 1/2(\xi^2+\xi_i\xi)(1-\eta^2); \rightarrow i=6,8 \\
N_i &= (1-\xi^2)(1-\eta^2); \rightarrow i=9.
\end{aligned}
\tag{11}
$$

Fig. 4 shows the discretization of a laminated composite beam.

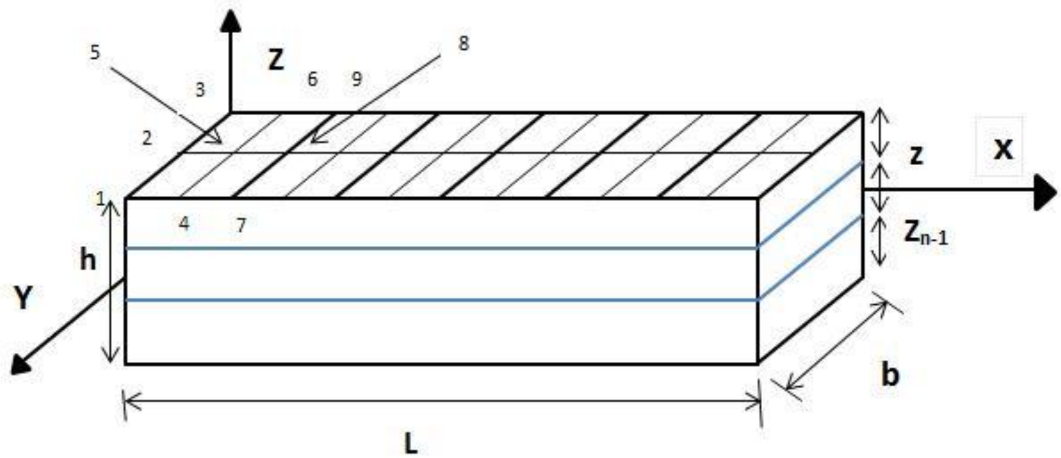


Fig. 4 Discretized composite beam.

3.5 Mass Matrix

The element mass matrix of the laminated composite beam element is

$$[M_e] = \int_{A_e} [N]^{'}[\rho][N]dA \tag{12}$$

Where [**ρ**] matrix for 1st order shear deformation theory is

$$[\rho] = \begin{bmatrix} I & 0 & P & 0 & 0 \\ 0 & I & 0 & 0 & P \\ 0 & 0 & I & 0 & 0 \\ P & 0 & 0 & Q & 0 \\ 0 & P & 0 & 0 & Q \end{bmatrix}. \tag{13}$$

Here I, P and Q are

$$(I, P, Q) = \int_{-h/2}^{h/2} \rho(z).(1, z, z^2)dz\,. \tag{14}$$

3.6 Element stiffness matrix

The element stiffness matrix for the beam element considered is

$$[K_e] = \int_{A_e} [B]^{'} [D][B] dA. \tag{15}$$

By assembling all the element mass and stiffness matrices with respect to the global co-ordinates, the following free vibration equation is obtained

$$[M]\{\ddot{d}\} + [K]\{d\} = 0. \tag{16}$$

Where {d} is the displacement vector. For integration of stiffness and mass matrix Gauss-quadrature method is used. The method proposed in the analysis is described in Appendix B.

3.7 Hamilton's principle

The Hamilton Principle may be expressed in the form [33]

$$\int_{t_0}^{t_1} (\delta T - \delta U + \delta V) dt = 0. \tag{17}$$

Where,

T – Kinetic energy.

U – Potential or Strain energy.

V – Work done by external forces.

δ – Represents the variation of T, U and V.

For a small element dm change in kinetic, potential or strain energy and work done are given as

$$dT = \frac{1}{2} dm\dot{y}^2, \qquad dU = \frac{M^2}{2EI} dx \qquad \text{and} \qquad dV = (Pdx)y \quad . \tag{18}$$

Integrating these for whole length of the beam gives

$$\delta T = \int_0^l \frac{W}{g} \dot{y} \delta \dot{y} dx, \qquad \delta U = \int_0^l EI \ddot{y} \delta \ddot{y} dx \qquad \text{and} \qquad \delta V = \int_0^l P \delta y dx \ . \tag{19}$$

Putting the values of δT, δU and δV from equation (19) into equation (17) one can obtain the equation of motion as

$$EI \frac{d^4 y}{dx^4} = P - \frac{W}{g} \ddot{y} \ . \tag{20}$$

$\frac{W}{g} \ddot{y}$ is a time dependent term. If harmonic motion is assumed then $\ddot{y} = -\omega^2 y$. So for mode shapes, general equation of motion becomes

$$EI \frac{d^4 y}{dx^4} - \frac{W}{g} \omega^2 y = 0. \tag{21}$$

For the vibration of beam, the general equation becomes

$$\frac{d^4 y}{dx^4} - \beta^4 y = 0 . \tag{22}$$

Where $\beta^4 = \frac{W \omega^2}{gEI}$. (23)

And its general solution is

$$y = A \cosh \beta x + B \sinh \beta x + C \cos \beta x + D \sin \beta x . \tag{24}$$

When this equation is solved for simple supported boundary condition then final solution is

$$y(x,t)=\sum_{n=1}^{\infty}(A\cos\omega_n t+B\sin\omega_n t)\sin\frac{n\pi x}{l}\ . \qquad (25)$$

This equation is used to find the mode shapes of the beam. Here ω_n is the natural frequency for a given mode shape.

FINITE ELEMENT MODEL FOR HSDT

4.1 Higher order shear deformation theory

In order to approximate a 3D elastic problem to a 2D beam problem the displacement functions (u, v, w) at a point (x, y, z) are expanded in Taylor series in terms of thickness co-ordinate [1]. As transverse shear stress vary parabolically throughout the beam thickness. So the displacement fields can be expanded to cubic power of thickness co-ordinate as follows

$$\begin{aligned} u(x,y,z) &= u_0(x,y) + z.\theta_x(x,y) + z^2.\phi_x(x,y) + z^3.\xi_x(x,y) \\ u(x,y,z) &= v_0(x,y) + z.\theta_y(x,y) + z^2.\phi_y(x,y) + z^3.\xi_y(x,y)\ . \\ w(x,y,z) &= w_0(x,y) + z.\theta_z(x,y) + z^2.\phi_z(x,y) + z^3.\xi_z(x,y) \end{aligned} \tag{26}$$

The strain displacement relations are [24, 29]

$$\begin{aligned} \varepsilon_x &= \varepsilon_{x_0} + z\kappa_x + z^2\varepsilon^*_{x_0} + z^3\kappa^*_x \\ \varepsilon_y &= \varepsilon_{y_0} + z\kappa_y + z^2\varepsilon^*_{y_0} + z^3\kappa^*_y \\ \varepsilon_z &= \varepsilon_{z_0} + z\kappa_z + z^2\varepsilon^*_{z_0} + z^3\kappa^*_z \\ \gamma_{xy} &= \varepsilon_{xy_0} + z\kappa_{xy} + z^2\varepsilon^*_{xy_0} + z^3\kappa^*_{xy} \\ \gamma_{yz} &= \varepsilon_{yz_0} + z\kappa_{yz} + z^2\varepsilon^*_{yz_0} + z^3\kappa^*_{xy} \\ \gamma_{xz} &= \varepsilon_{xz_0} + z\kappa_{xz} + z^2\varepsilon^*_{xz_0} + z^3\kappa^*_{xz}\ . \end{aligned} \tag{27}$$

Where

$$\varepsilon_{x_0} = \frac{\partial u_0}{\partial x}, \qquad \kappa_x = \frac{\partial \theta_x}{\partial x}, \qquad \varepsilon^*_{x_0} = \frac{\partial \phi_x}{\partial x}, \qquad \kappa^*_x = \frac{\partial \xi_x}{\partial x},$$

$$\varepsilon_{y_0} = \frac{\partial v_0}{\partial y}, \qquad \kappa_y = \frac{\partial \theta_y}{\partial y}, \qquad \varepsilon^*_{y_0} = \frac{\partial \phi_y}{\partial y}, \qquad \kappa^*_y = \frac{\partial \xi_y}{\partial y},$$

$$\varepsilon_{z_0} = \theta_z, \qquad \kappa_z = 2\phi_z, \qquad \varepsilon^*_{z_0} = 3\xi_z, \qquad \kappa^*_z = 0,$$

$$\varepsilon_{xy_0} = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}, \qquad \kappa_{xy} = \frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x}, \qquad \varepsilon^*_{xy_0} = \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x},$$

$$\kappa^*_{xy} = \frac{\partial \xi_x}{\partial y} + \frac{\partial \xi_y}{\partial x}, \qquad \varepsilon_{yz_0} = \theta_y + \frac{\partial w_0}{\partial y}, \qquad \kappa_{yz} = 2\phi_y + \frac{\partial \theta_z}{\partial y},$$

$$\varepsilon^*_{yz_0} = 3\xi_y + \frac{\partial \phi_z}{\partial y}, \qquad \kappa^*_{yz} = \frac{\partial \xi_z}{\partial y}, \qquad \varepsilon_{xz_0} = \theta_x + \frac{\partial w_0}{\partial x},$$

$$\kappa_{xz} = 2\phi_x + \frac{\partial \theta_z}{\partial x}, \qquad \varepsilon^*_{xz_0} = 3\xi_x + \frac{\partial \phi_z}{\partial y}, \qquad \kappa^*_{xz} = \frac{\partial \xi_z}{\partial x}. \tag{28}$$

These are in-plane strains and curvatures. The parameters φ_x, φ_y, φ_z, ξ_x, ξ_y, ξ_z are the higher order terms in the Taylor series expansion. They represent higher-order transverse cross-sectional deformation modes.

The stress-strain relation for a lamina with respect to the fiber-matrix co-ordinate axis (1, 2, 3) can be given as

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{12} \\ \tau_{23} \\ \tau_{13} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{13} \end{Bmatrix} . \qquad (29)$$

Here (σ_1, σ_2, σ_3, τ_{12}, τ_{23}, τ_{31}) are the stresses and (ε_1, ε_2, ε_3, γ_{12}, γ_{23}, γ_{13}) are the strain components corresponding to the lamina co-ordinates (1, 2, 3). C_{ij} is the compliance matrix w.r.t lamina axis (1, 2, 3) and is defined in appendix A. In laminate co-ordinates (x, y, z) the stress-strain relations for the lamina are

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & Q_{14} & 0 & 0 \\ Q_{12} & Q_{22} & Q_{23} & Q_{24} & 0 & 0 \\ Q_{13} & Q_{23} & Q_{33} & Q_{34} & 0 & 0 \\ Q_{14} & Q_{24} & Q_{34} & Q_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & Q_{55} & Q_{56} \\ 0 & 0 & 0 & 0 & Q_{56} & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} . \qquad (30)$$

Here (σ_x, σ_y, σ_z, τ_{xy}, τ_{yz}, τ_{zx}) are the stresses and (ε_x, ε_y, ε_z, γ_{xy}, γ_{yz}, γ_{zx}) are the strain components corresponding to the laminate co-ordinates (x, y, z). Q_{ij}'s are the transformed elasticity constants and are defined in appendix A.

From here the elasticity matrix [D] can be derived as

$$[D]=[T]^{'}[Q_{ij}][T]. \tag{31}$$

Here [T] is the thickness co-ordinate matrix. So, D-matrix for HSDT is

$$[D]=\begin{bmatrix} A_{ij} & B_{ij} & D_{ij} & E_{ij} & 0 & 0 & 0 & 0 \\ B_{ij} & D_{ij} & E_{ij} & F_{ij} & 0 & 0 & 0 & 0 \\ D_{ij} & E_{ij} & F_{ij} & G_{ij} & 0 & 0 & 0 & 0 \\ E_{ij} & F_{ij} & G_{ij} & H_{ij} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & AA_{ij} & BB_{ij} & DD_{ij} & EE_{ij} \\ 0 & 0 & 0 & 0 & BB_{ij} & DD_{ij} & EE_{ij} & FF_{ij} \\ 0 & 0 & 0 & 0 & DD_{ij} & EE_{ij} & FF_{ij} & GG_{ij} \\ 0 & 0 & 0 & 0 & EE_{ij} & FF_{ij} & GG_{ij} & HH_{ij} \end{bmatrix}. \tag{32}$$

Here

$$\left(A_{ij},B_{ij},D_{ij},E_{ij},F_{ij},G_{ij},H_{ij}\right)=\int_{-h/2}^{h/2} Q_{ij}\left(1,Z,Z^2,Z^3,Z^4,Z^5,Z^6\right)dZ \rightarrow i,\, j=1,2,3,4 \tag{33}$$

And

$$\left(AA_{ij},BB_{ij},DD_{ij},EE_{ij},FF_{ij},GG_{ij},HH_{ij}\right)=\int_{-h/2}^{h/2} Q_{ij}\left(1,Z,Z^2,Z^3,Z^4,Z^5,Z^6\right)dZ \rightarrow i,\, j=5,6.$$

Here [A_{ij}] is extensional stiffness matrix, [B_{ij}] is stretching-bending coupling matrix and [D_{ij}] is flexural stiffness matrix.

3.2 Strain-Displacement Matrix

The strain-displacement relation for the laminate is

$$\{\varepsilon\} = [L]\{\delta\}. \tag{34}$$

Here [L] is operator matrix, {ε} is the in-plane strain matrix and {δ} is the displacement at any point on the mid-plane of the element. The strain-displacement matrix [B] is obtained by multiplying operator matrix by shape functions as

$$[B] = [L] * N_r \rightarrow r = 1, 2, 3, 4...9. \tag{35}$$

The B-matrix for FSDT has been written in Appendix C.

Nine-noded Isoparametric element is used for this theory. This has been explained earlier in chapter 2.

3.3 Mass Matrix

The element mass matrix is [1]

$$[M_e] = \int_{A_e} [N]^{'}[\rho][N]dA. \tag{36}$$

Where [**ρ**] is the mass matrix taking account for thickness co-ordinate in the laminate and [N] is the shape function matrix.

Here [**ρ**] matrix for higher order shear deformation theory is

$$[\rho]=\begin{bmatrix} I&0&0&P&0&0&Q&0&0&R&0&0\\ 0&I&0&0&P&0&0&Q&0&0&R&0\\ 0&0&I&0&0&P&0&0&Q&0&0&R\\ P&0&0&Q&0&0&R&0&0&S&0&0\\ 0&P&0&0&Q&0&0&R&0&0&S&0\\ 0&0&P&0&0&Q&0&0&R&0&0&S\\ Q&0&0&R&0&0&S&0&0&T&0&0\\ 0&Q&0&0&R&0&0&S&0&0&T&0\\ 0&0&Q&0&0&R&0&0&S&0&0&T\\ R&0&0&S&0&0&T&0&0&U&0&0\\ 0&R&0&0&S&0&0&T&0&0&U&0\\ 0&0&R&0&0&S&0&0&T&0&0&U \end{bmatrix}. \qquad (37)$$

Here I, P, Q, R, S, T and U are as

$$(I,P,Q,R,S,T,U)=\int_{-h/2}^{h/2}\rho(z)\times(1,z,z^2,z^3,z^4,z^5,z^6)dz. \qquad (38)$$

3.4 Element Stiffness Matrix

The element stiffness matrix for the beam element considered is obtained as [30]

$$[K_e]=\int_{A_e}[B]^{'}[D][B]dA. \qquad (39)$$

By assembling all the element mass and stiffness matrices with respect to the global co-ordinates, the following free vibration equation can be given as

$$[M]\{\ddot{d}\}+[K]\{d\}=0. \qquad (40)$$

Where {d} is the displacement vector. For integration of stiffness and mass matrix Gauss-quadrature method is used. This method is described in Appendix B. Hamilton principle is used to find mode shapes. Hamilton principle is described in the previous chapter.

RESULT AND DISCUSSION

5.1 Numerical analysis

For both first and higher order shear deformation theories MATLAB program is written [27]. Eigen value analysis is used to find natural frequencies for the laminated composite beam. Also an ANSYS, APDL program is written for the same laminated beam. The results of all the three programs are compared for different boundary conditions, (l/h) ratios and different stacking sequences.

The numerical results are obtained for free vibration of composite beams using FSDT, HSDT and ANSYS. The shear correction factor is assumed to be 5/6 for the first-order theory. The lamina properties used are as follows

E1=129.207 GPa; E2=E3=9.42512 GPa; G12=5.15658 GPa; G13=4.3053 GPa;

G23=2.5414GPa; ν_{12}=0.3; ν_{23}=0.218; ν_{21}=0.021; $\boldsymbol{\rho}$=1550.0666kg/m^3

The beam is assumed to have a length of 0.1905 m and a width of 0.0127 m. The following boundary conditions are used

For simply supported end condition

$$u_0 = v_0 = w_0 = \theta_x = \phi_x = \phi_y = \phi_z = \xi_x = 0\,.$$

For clamped supported end condition

$$u_0 = v_0 = w_0 = \theta_x = \theta_y = \theta_z = \phi_x = \phi_y = \phi_z = \xi_x = \xi_y = \xi_z = 0.$$

For free edge end condition

$$u_0, v_0, w_0, \theta_x, \theta_y, \theta_z, \phi_x, \phi_y, \phi_z, \xi_x, \xi_y, \xi_z.$$

have not been specified.

METHODOLOGY

FINDING OUT MATERIAL PROPERTIES USING RULE OF MIXTURE

MAKE STRAIN-DISPLACEMENT RELATIONS

MAKE STRESS-STRAIN RELATION

TRANSFORM THESE STRESSES FROM LOCAL TO GLOBAL CO-ORDINATES

FIND OUT LOAD-DEFLECTION AND MOMENT-CURVATURE RELATION

USE BOUNDARY CONDITIONS AND FIND ‘D’ MATRIX

FIND ‘B’ MATRIX USING SHAPE FUNCTIONS

FIND STIFFNESS MATRIX $\boldsymbol{K} = \iint_{0}^{L} [\boldsymbol{B}]^{T} [\boldsymbol{D}][\boldsymbol{B}]\, \boldsymbol{da}$

FIND MASS MATRIX $\boldsymbol{M} = \iint_{0}^{L} [\boldsymbol{N}]^{T} [\boldsymbol{\rho}][\boldsymbol{N}]\, \boldsymbol{da}$

FIND OUT NATURAL FREQUENCIES, MODE-SHAPES FOR DIFFERENT END SUPPORTS CONDITIONS USING EIGEN VALUE ANALYSIS

5.2 Results and discussion

The non-dimensional natural frequencies λ for different boundary conditions of unidirectional composite beam varies for various (l/h) ratios. The present results are found to be in good agreement with those of Maity [1] and there is no appreciable discrepancy between the first order and higher order theory in the case of clamped-free uniaxial beams varying from thin (l/h=60) to thick (l/h=5) laminates. The results reveal the usual trends. The frequency decreases when the fiber is oriented from 0^0 to 90^0. Table 1 provides the value λ for uniaxial composite beams having clamped free end condition. In addition, there do not exist wide discrepancies between the first-order theory and the higher-order theory. Furthermore, the frequencies are found to reduce with the increase in fiber orientation. Tables 2 illustrate the non-dimensional fundamental frequencies for laminated composite beams with different support conditions and different stacking sequences for (1/h=60). Practically no appreciable difference is observed employing the first-order theory, higher-order theory and ANSYS for the thin *(*1/h=60) laminated composite beams. For large (l/h) ratios there are low transverse shear stresses. So this reduces the effect of shear correction factor in first order shear deformation theory. So this results in approximately equal values of natural frequencies from each theory.

Table 3 shows variation of non-dimensional natural frequency with different stacking sequences for (l/h=5). The results of HSDT, FSDT and ANSYS are compared in the table provided for all the three boundary conditions. The results of both theories are found in good agreement with each other and ANSYS results too for clamped free end condition.

However, some discrepancies are observed for the case of thick (l/h=5) laminated composite beams, especially when the end conditions are clamped and simply supported. This is because an increase in thickness of the lamina increases the transverse shear stresses.

Table 1

Non Dimensional Fundamental Frequencies ($\lambda=((\rho A/E_3 I)^{1/2}L^2\omega)$ for Clamped-free unidirectional composite beam

Method	Θ=0	Θ=30	Θ=45	Θ=60	Θ=90
l/h=60					
HSDT	12.9941	5.3908	4.1808	3.7060	3.5155
FSDT	12.9940	5.3926	4.1815	3.7061	3.5154
Ref	12.8029	5.3169	4.0919	3.7184	3.6421
l/h=20					
HSDT	12.7947	5.3462	4.1632	3.6967	3.5079
FSDT	12.6450	5.3374	4.1627	3.6981	3.5103
l/h=10					
HSDT	8.4935	3.5333	2.7722	2.4695	2.3460
FSDT	8.4913	3.5301	2.7678	2.4638	2.3386
l/h=5					
HSDT	4.2485	1.7690	1.3895	1.2395	1.1794
FSDT	4.2456	1.7651	1.3839	1.2319	1.1693

Table 2

Non Dimensional Fundamental Frequencies ($\lambda = ((\boldsymbol{\rho}A/E_3I)^{1/2}L^2\omega)$ for composite beam (l/h=60)

Lamination	**C.F**	**C.C**	**S.S**
Symmetric	*HSDT FSDT ANSYS*	*HSDT FSDT ANSYS*	*HSDT FSDT ANSYS*
0/30/0	12.854 12.861 12.84	80.691 80.753 80.285	36.048 36.074 36.029
0/45/0	12.788 12.794 12.76	80.179 80.282 79.735	35.846 35.869 35.815
0/60/0	12.767 12.769 12.73	79.928 80.065 79.444	35.777 35.791 35.732
0/90/0	12.767 12.778 12.72	79.817 80.047 79.377	35.771 35.812 35.749

C.F. Clamped-free; C.C, Clamped-Clamped, S.S, Simply-Supported

Table 3

Non Dimensional Fundamental Frequencies ($\lambda = ((\boldsymbol{\rho}A/E_3I)^{1/2}L^2\omega)$ for composite beams (l/h=5)

Lamination	**C.F**	**C.C**	**S.S**
Symmetric	*HSDT FSDT ANSYS*	*HSDT FSDT ANSYS*	*HSDT FSDT ANSYS*
0/30/0	3.662 3.663 3.785	20.937 22.062 22.289	19.482 22.062 22.327
0/45/0	3.588 3.617 3.662	20.455 21.463 21.558	19.105 21.463 21.351
0/60/0	3.562 3.575 3.611	20.146 20.831 20.790	18.706 20.831 20.858
0/90/0	3.552 3.560 3.587	19.843 20.065 19.926	18.249 20.065 19.989

C.F. Clamped-free; C.C, Clamped-Clamped, S.S, Simply-Supported

So the shear correction factor comes in action for first order shear deformation theory. Also both sided end support conditions (CC and SS) results in large shear. This results in discrepancy in results for each theory. This difference in the non-dimensional fundamental frequency is, however, restricted to within 10%.

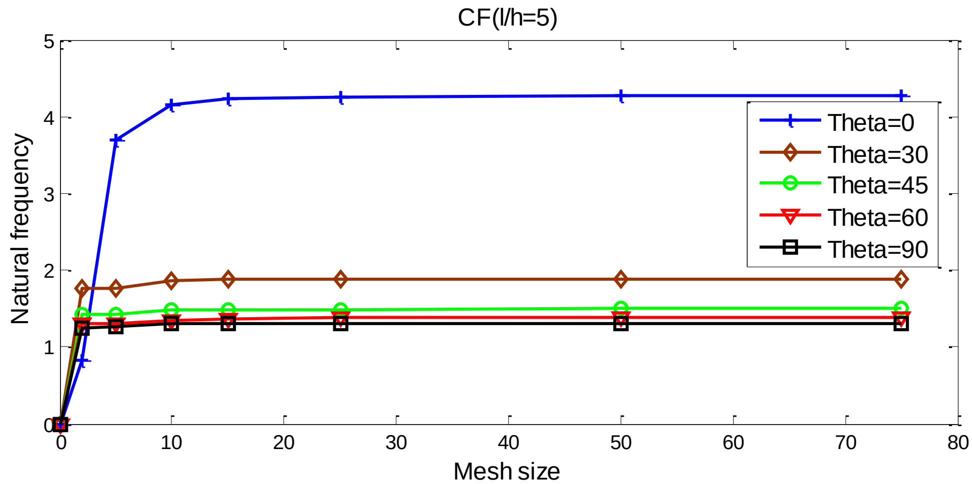


Fig. 5 Natural Frequency for diff. values of theta and constant (l/h=5).

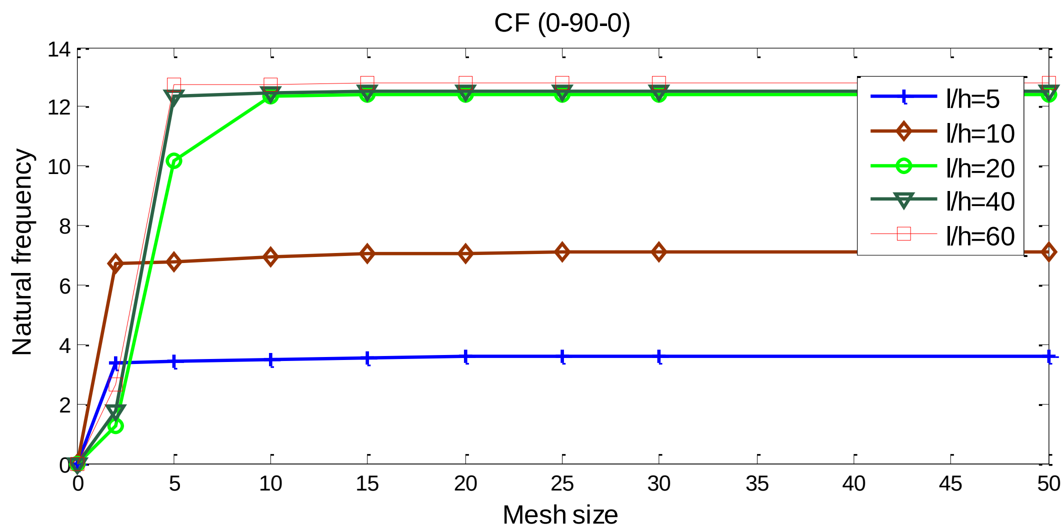


Fig. 6 Natural Frequency for different l/h ratio for (0/90/0).

From Fig. 5 one can observe that by increasing the ply angle for the lamina from 0 to 90, natural frequency decreases and how they converge to their respective values for constant (l/h=5) ratio. This is explained as when ply angle changes from 0 to 90, transverse elasticity modulus accounts more for stiffness of the beam, and elasticity modulus of transverse direction is much smaller than elasticity modulus of longitudinal direction, and stiffness of the material is directly proportional to elasticity modulus. So when stiffness decreases with the same weight and geometry, correspondingly it decreases the natural frequency of the material.

Fig. 6 shows that by increasing the (l/h) ratio from 5 to 60 for the lamina configuration (0-90-0) the natural frequency decreases. This happens because on decreasing the height of the laminate beam by maintaining the same stacking sequence, natural frequency increases as a result of decrease in moment of inertia of the cross-section, and λ is inversely proportional to moment of inertia as shown above. An increase in thickness of the co-ordinate increases damping, which decreases non dimensional natural frequency. Thus by increasing l/h ratio, there is an increase in natural frequency of the beam.

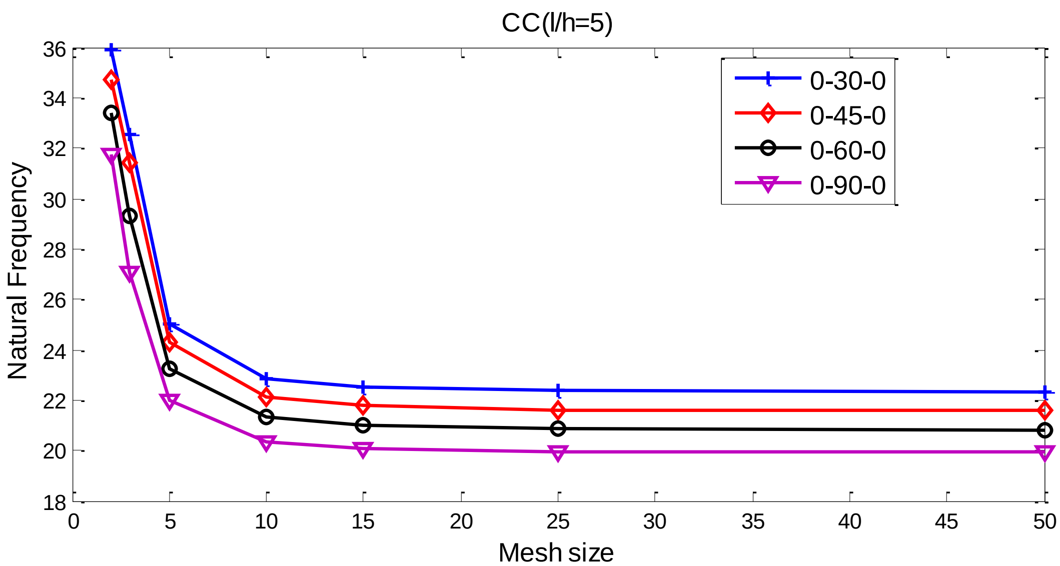


Fig. 7 Natural Frequency for different stacking sequence for (l/h=5) and CC boundary condition.

Fig. 7 shows variation of natural frequency with different stacking sequence in a laminate for (l/h) ratio of 5 and clamped-clamped boundary condition. The plot clearly depicts that as we increase the middle ply angle of the laminate its natural frequency decreases. Thus increase in ply angle of a laminate decreases its natural frequency.

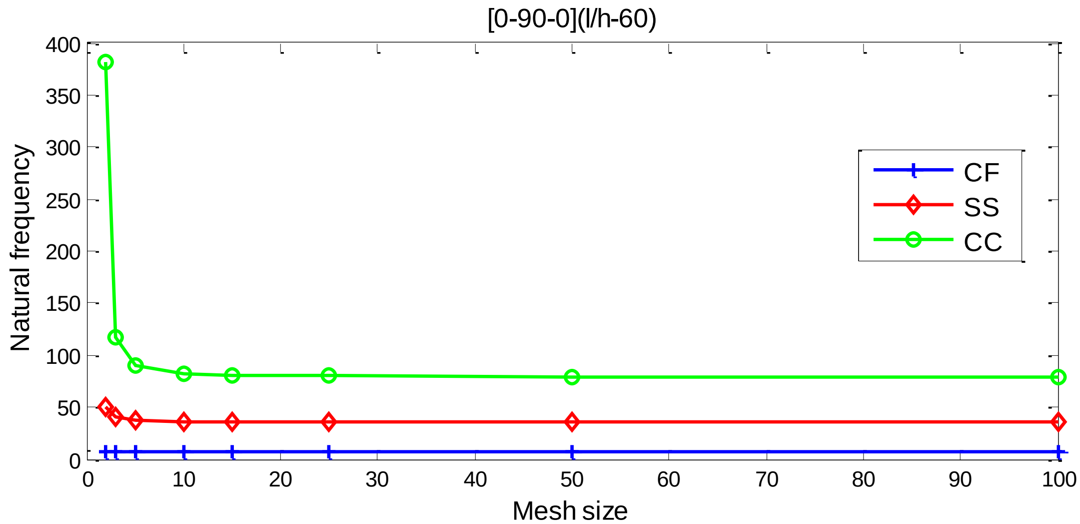


Fig. 8 Natural Frequency for different boundary conditions for (0/90/0) and (l/h=60).

Fig. 8 shows variation of natural frequency with three boundary conditions for a laminate having stacking sequence of (0-90-0). The laminate has (l/h) ratio of 60. The plot clearly depicts that boundary condition has a huge impact on natural frequency of the beam. Clamped-free has the lowest non-dimensional natural frequency while clamped-clamped has highest non-dimensional natural frequency. Simply supported LCB has a 200 % higher frequency when compared to clamped free LCB and clamped-clamped LCB has a 560 % higher frequency when compared to clamped free LCB for (l/h=60) and stacking sequence (0-90-0).

Static analysis in ANSYS finite element software is also computed [25, 26]. An APDL program is written, where shell 181 element is used. It is a 2D element with four corner nodes and each node having six DOF's. As mesh size increases, natural frequency obtained converges. Block-Lanczos method is used to find natural frequencies and mode shapes in ANSYS. This is a numerical method for solving Eigen value problems. The Block-Lanczos method uses the sparse matrix results in less memory and time usage. The benefit of using block-lanczos method is having same convergence rate while extracting modes in mid-range or in higher range.

First four mode shape obtained for (l/h=60), stacking sequence (0-90-0) and clamped-clamped or simply-supported boundary condition from ANSYS are

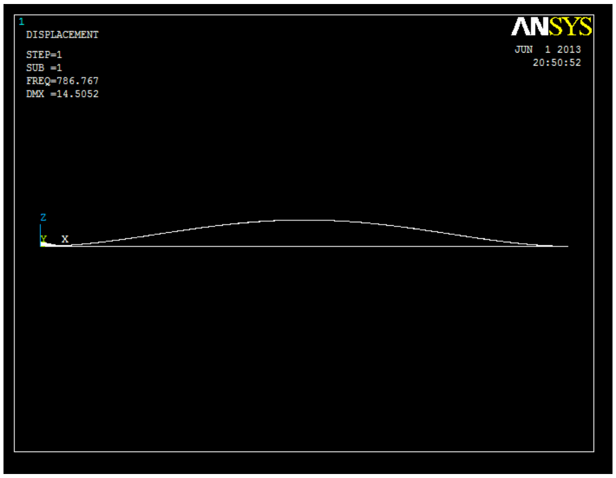


Fig. 9 1st mode shape for clamped-clamped BC.

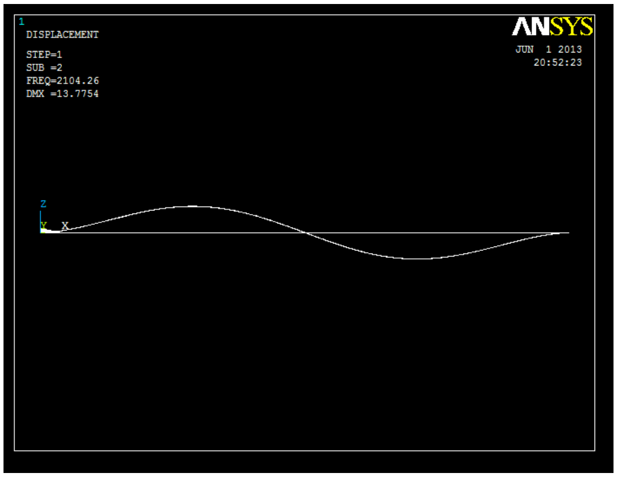


Fig. 10 2nd mode shape for clamped-clamped BC.

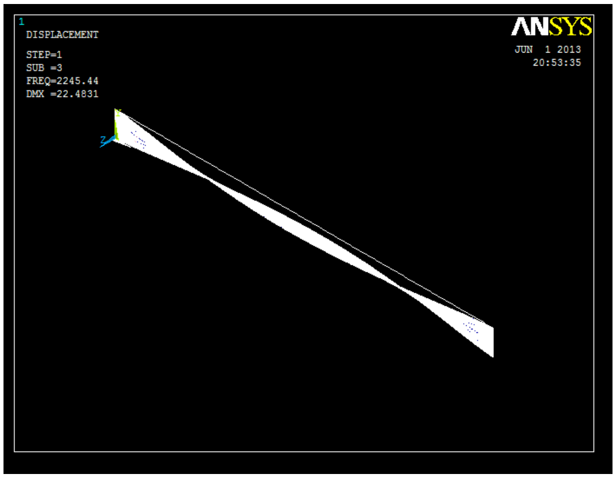


Fig. 11 3rd mode shape for clamped-clamped BC.

Fig. 12 4th mode shape for clamped-clamped BC.

Here 1st and 2nd mode are vertical bending mode, 3rd mode is torsional mode and 4th mode is lateral bending mode of vibration.

First four mode shape obtained for (l/h=60), stacking sequence (0-90-0) and clamped-free boundary condition from ANSYS are

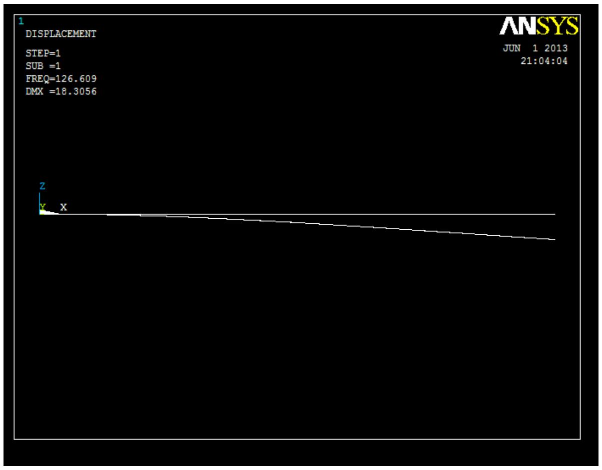


Fig. 13 1st mode shape for clamped-free BC.

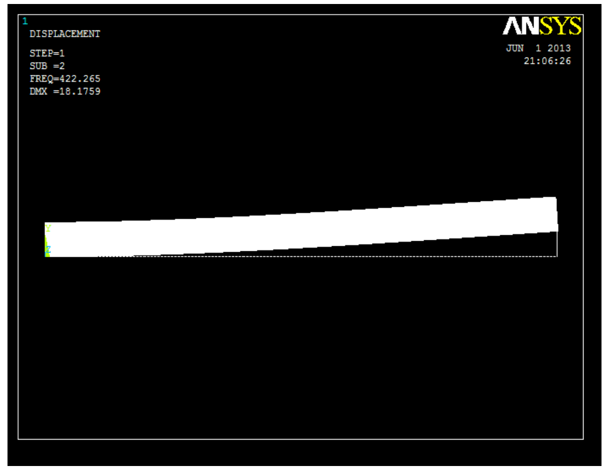
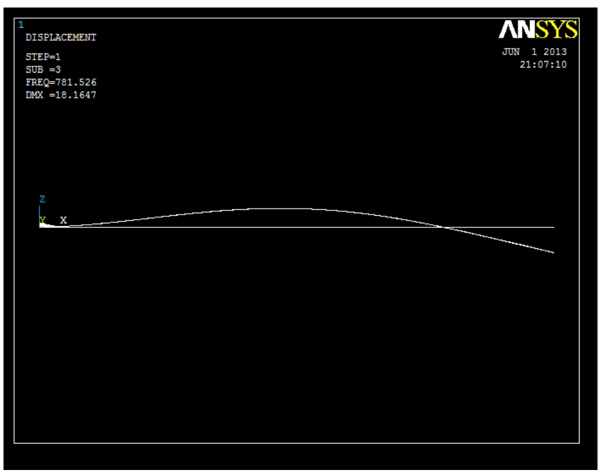


Fig. 14 2nd mode shape for clamped-free BC.



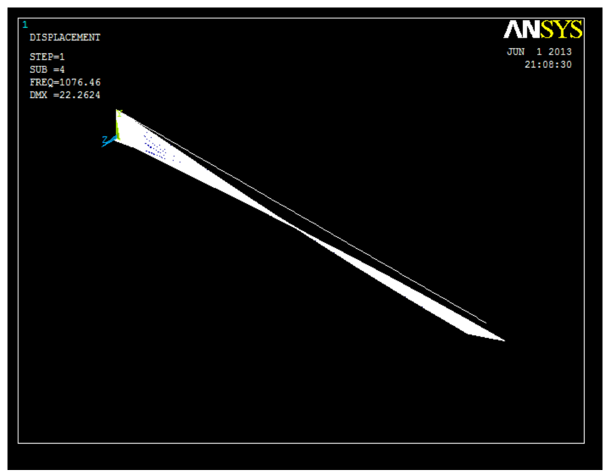


Fig. 15 3rd mode shape for clamped-free BC.

Fig. 16 4th mode shape for clamped-free BC.

Here 1st and 3rd mode are vertical bending mode, 2nd mode is lateral bending mode and 4th mode is torsional mode of vibration.

CONCLUSIONS AND FUTURE WORK

A finite element analysis method used to study the free vibration characteristics of laminated composite beams has been developed using a higher-order shear deformation theory and the conventional first-order shear deformation theory. Nine-noded isoparametric elements are used to discretize the analysis domain. The present results also compare well with those of Maiti [1] and with the ANSYS program written for the problem described. Therefore, the present higher-order theory is expected to yield accurate estimation of frequencies. It is observed that there is not much difference in frequencies using the higher-order and first-order shear deformation theories. As ply angle of the lamina increases from 0 to 90, natural frequency decreases. As (l/h) ratio of the beam increases from 5 to 60, the non-dimensional fundamental frequency increases. Boundary conditions have a huge impact on natural frequency of the beam. Small (l/h) ratio causes discrepancy in computation of non-dimensional fundamental frequencies for clamped-clamped and simply supported conditions, so a 3D finite element model can also be derived for this problem. Bending analysis can be done in future for the same laminated composite beam.

APPENDIX

Appendix A

When m = cos θ and n = sin θ. The transformed [Q] matrix co-efficients are

$$Q_{11} = C_{11}m^4 + 2\left(C_{12} + 2C_{44}\right)m^2n^2 + C_{22}n^4,$$

$$Q_{12} = C_{12}\left(m^4 + n^4\right) + \left(C_{11} + C_{22} - 4C_{44}\right)m^2n^2,$$

$$Q_{13} = C_{13}m^2 + C_{23}n^2,$$

$$Q_{14} = \left(C_{11} - C_{22} - 2C_{44}\right)m^3n + \left(C_{11} - C_{22} + 2C_{44}\right)mn^3,$$

$$Q_{22} = C_{11}n^4 + 2\left(C_{12} + 4C_{44}\right)m^2n^2 + C_{22}m^4,$$

$$Q_{23} = C_{13}n^2 + C_{23}m^2,$$

$$Q_{24} = \left(C_{11} - C_{22} - 2C_{44}\right)mn^3 + \left(C_{11} - C_{22} + 2C_{44}\right)m^3n,$$

$$Q_{33} = C_{33},$$

$$Q_{34} = \left(C_{31} - C_{32}\right)mn,$$

$$Q_{44} = \left(C_{11} - 2C_{12} + C_{22} - 2C_{44}\right)m^2n^2 + C_{44}\left(m^4 + n^4\right),$$

$$Q_{55} = C_{55}m^2 + C_{66}n^2,$$

$$Q_{56} = \left(C_{66} - C_{55}\right) mn,$$

$$Q_{66} = C_{55} n^2 + C_{66} m^2.$$

And compliance matrix co-efficients are

$$C_{11} = \frac{E_1(1-\nu_{23}\nu_{32})}{\Delta}, \qquad C_{12} = \frac{E_1(\nu_{21}+\nu_{31}\nu_{23})}{\Delta}, \qquad C_{13} = \frac{E_1(\nu_{31}+\nu_{21}\nu_{32})}{\Delta},$$

$$C_{22} = \frac{E_2(1-\nu_{13}\nu_{31})}{\Delta}, \qquad C_{23} = \frac{E_2(\nu_{32}+\nu_{12}\nu_{31})}{\Delta}, \qquad C_{33} = \frac{E_3(1-\nu_{12}\nu_{21})}{\Delta},$$

$$C_{44} = G_{12}, \qquad C_{55} = G_{23}, \qquad C_{66} = G_{13}.$$

Appendix B - Gauss-Quadrature method

Gauss-Legendre quadrature is very useful for integration of polynomial functions. It can integrate a polynomial function of order (2n-1) by using the n points exactly.

In numerical integration if we take a finite number of calculations. So the integral is approximated to

$$\int_a^b f(x)dx = \sum_{i=1}^{M} f(x_i) W_i.$$

Where

M – No. of integration points.

X_i – Integration point or Sampling point.

W_i – Weighing co-efficient.

Weighing co-efficients can be interpreted as the width of the rectangular strip whose height is $f(x_i)$. Any numerical integration may be expressed in this form. In order to derive standard values for the integration points and weighting co-efficients, the integration domain is normalized such that -1 ≤ x ≤ -1.

So, stiffness and mass matrix are integrated as

$$[K_e] = \int_{-1}^{1}\int_{-1}^{1} [B]^{'}[D][B]\left|J\right| d\xi d\eta,$$

$$\left[M_e\right] = \int_{-1}^{1}\int_{-1}^{1} \left[N\right]^{'}\left[\rho\right]\left[N\right]\left|J\right| d\xi d\eta.$$

Here |J| is jacobian matrix and is defined as

$$\left[J\right] = \begin{bmatrix} \frac{dx}{d\xi} & \frac{dy}{d\xi} \\ \frac{dx}{d\eta} & \frac{dy}{d\eta} \end{bmatrix}.$$

Here x and y aye generalized co-ordinates while ξ and η are natural co-ordinates.

Appendix C – B-matrix for HSDT

$$
B=\begin{bmatrix}
\frac{\partial N_r}{\partial x} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{\partial N_r}{\partial y} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & N_r & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{\partial N_r}{\partial y} & \frac{\partial N_r}{\partial x} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{\partial N_r}{\partial x} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{\partial N_r}{\partial y} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2N_r & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{\partial N_r}{\partial y} & \frac{\partial N_r}{\partial x} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial N_r}{\partial x} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial N_r}{\partial y} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3N_r \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial N_r}{\partial y} & \frac{\partial N_r}{\partial x} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial N_r}{\partial x} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial N_r}{\partial y} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial N_r}{\partial y} & \frac{\partial N_r}{\partial x} & 0 \\
0 & 0 & \frac{\partial N_r}{\partial y} & 0 & N_r & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{\partial N_r}{\partial x} & N_r & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{\partial N_r}{\partial y} & 0 & 2N_r & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{\partial N_r}{\partial x} & 2N_r & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial N_r}{\partial y} & 0 & 3N_r & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial N_r}{\partial x} & 3N_r & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial N_r}{\partial y} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial N_r}{\partial x}
\end{bmatrix}
$$

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